from this that $\sigma(t, \varphi(t)) = \sigma_0(t)$, whence the assertion made above also follows). A direct derivation of Eq.(5.3) has been given in /1/ where it was also shown that this equation is integrated in quadratures and a functional relationship is obtained which relates $\sigma = \sigma(t, \varphi(t))$, $\sigma_0(t)$ and t.

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THE RELATIONSHIP BETWEEN THE ENDOCHRONIC THEORY OF PLASTICITY AND THE "NEW" MEASURE OF INTERNAL TIME*

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The transition from an earlier version of the endochronic theory of plasticity (ETP) to a theory with a "new" measure of internal time is considered together with the mutual relationship between the latter and flow theory.

The endochronic theory of plasticity was initially put forward as a theory in which there was no yield surface (YS) /1/. This was its essential difference both from the well-known classical theories of plasticity (of the flow-theory type) and from the many modern theories which are based on the concept of a yield surface. Recently, however, a version of the isochronic theory of plasticity has become widely used which is based on a "new" measure of the internal time for which the Odqvist parameter /2-4/ is actually used. There is already a yield surface in this version of the theory which may be considered as a rejection by the authors of this approach of the initial idea of constructing an analytical (non-singular) plasticity functional for arbitrary complex deformation processes.

1. We shall use a vector representation of the loading and deformation processes. Let σ and e be the stress and deformation vectors respectively /5/.

The ETP functional is written in the form

$$\sigma = \int_0^z J(z-\eta)\, de(\eta) \tag{1.1}$$

and is formally analogous to the linear viscoelasticity functional only, instead of the physical time, a new parameter z, referred to as the internal time /1/, is used to describe the history of the deformation and loading processes. Generally speaking, the internal time z is assumed to be a functional of the deformation process. Several possible definitions of this quantity have been proposed. It was initially thought that

$$dz = ds/f(s), \quad ds = |de| \tag{1.2}$$

where the function $f\ (f>0)$ is responsible for the effects of isotropic strengthening (or weakening) of the material and is usually called the strengthening function.

Refinement of the initial version of ETP proceeded in several directions. For instance, an alternative approach to the construction of certain ETP relationships was proposed in /6/. This approach was based on a more complex tensor-parametric form of writing the plasticity functional.

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Another route to the removal of the drawbacks of the initial version of ETP reduces /2, 3/ to the replacement of the internal time measure. In fact, instead of definition (2.2), it was proposed that one should use the new measure of internal time

$$dz = d\xi / f(\xi), \quad d\xi = |de - \chi E^{-1} d\sigma| \tag{1.3}$$

as the parameter of the history of the deformation and loading processes, where E is the modulus of elasticity and χ is an additional new parameter of the model. The physical meaning of this parameter was not discussed but it was assumed that $\chi \in [0,1]$. It should be noted that an analogous proposition had already been implicitly put forward in /7/ but the fact that no examples were given prevented any investigation of the advantages of the new approach. The introduction of a new measure of internal time enabled the description of experimental data within the framework of ETP to be improved considerably /8-9/*. (*These questions have been considered in the greatest detail in the paper: Mosolov A.B., Endochronic theory of plasticity, Preprint 353, Inst. Problem Mekhaniki, Akad. Nauk SSSR, Moscow, 1988.).

The measure (1.3) with $\chi = 1$ very rapidly became used in the majority of papers concerned with ETP. In essence, the Odqvist parameter (the plastic length of an arc) was therefore taken as the parameter which describes the history of the deformation process. This led to the need to consider singular kernels J and to introduce the concept of a yield surface. Correspondingly, the ETP functionals became written in the form /3/

$$\sigma = \int_0^z J(z-\eta)\, d\mathbf{e}_p(\eta), \quad J(z) = z^{-\alpha} J^{(0)}(z) \tag{1.4}$$

or in the form /4/

$$\sigma = \sigma_0 \frac{d\mathbf{e}_p}{dz} \int_0^z J^{(0)}(z-\eta)\, d\mathbf{e}_p(\eta) \tag{1.5}$$

where $\mathbf{e}_p$ is the vector of the plastic deformations, $J^{(0)}$ is a singular kernel, and σ_0 is the limiting flow.

It will be shown below that a limiting surface arises in the theory (1.1) and (1.3) as $\chi \to 1$ which may be identified with a classical yield surface.

An alternative way of removing the drawbacks of the earlier versions of enochronic theory has been demonstrated in /6, 10-12/. In particular, it was recommended in these papers that an internal time should be introduced according to the formula

$$dL = |de - (E^{-1} - \beta/(\sigma_0 + \beta k))\, d\sigma|$$

where σ_0 is the limiting flow, k is the strengthening factor and β is a small parameter.

It can be shown that the parameter L is close to the Odqvist parameter but it is not identical to it. The theory does not have a yield surface. The results obtained within the framwork of this plasticity theory can differ appreciably from the corresponding results obtained on the basis of flow theory. This is particularly so in the case of cyclic processes.

2. Let us now consider an ETP functional in the form of (1.1), (1.3) as $\chi = 1$ under the assumption that the kernel $J(z)$ is a non-singular, fairly smooth function which, for example, is doubly differentiable with respect to z. Since the plasticity functional necessarily satisfies the hysteresis property /5/, it may be assumed that $J'(z) < 0$ (here and subsequently, we shall denote an arbitrary function with respect to the corresponding argument by a prime).

On differentiating relationship (1.1), we write the ETP functional in the form

$$d\sigma = J_0 de + \int_0^z J'(z-\eta)\, de(\eta)\, dz, \quad J_0 = J(0) \tag{2.1}$$

It follows from this that the modulus of elasticity E must be identified with the quantity J_0. From (2.1), we get

$$\left| de - \frac{d\sigma}{J_0} \right| = \frac{1}{J_0} \left| \int_0^z J'(z-\eta)\, de(\eta) \right| \left| de - \frac{d\sigma}{J_0} \right| \frac{1}{f(\xi)}$$

It follows from this equality that either $dz = 0$ and Hooke's law $d\sigma = J_0 de$ is then satisfied or $dz \neq 0$ and the equality

$$\left| \int_0^z J'(z-\eta)\, de(\eta) \right| = J_0 f(\xi); \quad z = \int_0^\xi \frac{d\eta}{f(\eta)} = z(\xi), \quad d\xi = \left| de - \frac{d\sigma}{J_0} \right| \tag{2.2}$$

holds.

Actually, condition (2.2) now defines the limiting surface. Let us investigate this condition in greater detail.

Let us define the plastic deformations by the equality

$$de_p = de - J_0^{-1} d\sigma \tag{2.3}$$

On susbstituting (2.3) into (1.1) and integrating by parts, we have

$$-\int_0^z J'(z-\eta)\,\sigma(\eta)\,d\eta = J_0 \int_0^z J(z-\eta)\,de_p(\eta)$$

Differentiating this relationship with respect to z and passing to the limit as $z \to +0$, we obtain $(J_0' = J'(0))$

$$\sigma(+0) = -(J_0^2/J_0')\, f(0)\, de_p/d\xi\,|_{\xi\to+0}$$

The resulting relationship simply means that a limiting surface arises in ETP with $\chi = 1$ and that the initial position of this surface is described by the equation (we recall that $J'(z) \leqslant 0$)

$$\sigma_u = \sigma_0, \quad \sigma_0 = -(J_0^2/J_0')\, f(0)$$

Let us now consider the evolution of the limiting surface during the deformation process. In order to do this, we will make use of relationship (2.2). By substituting the quantity de_p, defined by (2.3) into this relationship and integrating by parts, we obtain

$$\left| \int_0^z J'(z-\eta)\,de_p(\eta) + \frac{1}{J_0}\int_0^z J''(z-\eta)\,\sigma(\eta)\,d\eta + \frac{J_0'}{J_c}\,\sigma(z) \right| = J_0 f(\xi) \tag{2.4}$$

We will now define a vector R by the equality

$$\mathbf{R}(z) = -\frac{J_0}{J_0'}\int_0^z J'(z-\eta)\,de_p(\eta) - \frac{1}{J_0'}\int_0^z J''(z-\eta)\,\sigma(\eta)\,d\eta \tag{2.5}$$

Relationship (2.4) can then be represented in the form

$$|\sigma - \mathbf{R}| = -(J_0^2/J_0')\, f(\xi) = \sigma_0(\xi) \tag{2.6}$$

We conclude from this that the limiting surface is a sphere of radius $\sigma_0(\xi)$, the centre of which is displaced to the point R. The function $f(\xi)$ defines the dimensions of the limiting surface and, in this sense, is actually a hardening function.

It may be verified that the vector R will be identically equal to zero only if $J(z) = Ee^{-\alpha z}$ In this case, the ETP functional is conveniently written in differential form which is identical to the three-term plasticity law /5/: $d\sigma = Ede - \alpha\sigma dz$. Such a model describes a material with isotropic strengthening.

When $\chi = 1$, the functional (1.1) can be written in the form of (1.4) which explicitly takes account of the presence of a yield surface. The kernels $J(z)$ and $J_0(z)$ are found to be associated by the relationship

$$J_0(z) + \frac{1}{J_0'}\int_0^z J''(z-\eta)\,J_0(\eta)\,d\eta = -\frac{J_0}{J_0'}\,J'(z) + \left(\frac{J_0}{J_0'}\right)^2 J''(z)$$

$$\sigma_0 = -J_0^2/J_0'$$

Let us now consider a simple example when the kernel J is taken in the form

$$J(z) = \mu + Ee^{-\alpha z}$$

We will show by direct calculations that, in this case

$$\sigma_0(\xi) = (E+\mu)^2 f(\xi)/(\alpha E), \quad R = \mu_1 e_p, \quad \mu_1 = \mu(E+\mu)/E$$

Consequently, a model of a material with a combination of transitional-isotropic hardening is obtained with such a kernel. The translational hardening is linear while the isotropic hardening is described by the function $f(\xi)$.

Using (2.1) and (2.6), the plasticity law can be rewritten in the form

$$de_p = [(\sigma - \mathbf{R})/\sigma_0(\xi)]\,d\xi \tag{2.7}$$

This equation holds only if the vector σ lies on the limiting surface. The matching conditions which are conventional in the theory of plasticity lead to a relationship which enables the plasticity law to be rewritten in the form

$$de_p = (\sigma - R)[\sigma_0^2(\xi)\sigma_0'(\xi)]^{-1}(\sigma - R)(d\sigma - dR)$$

By differentiating relationship (2.7) with respect to ξ, we obtain

$$\frac{d\sigma}{d\xi} = \sigma_0(\xi)\frac{d^2 e_p}{d\xi^2} + \frac{dR}{d\xi} + \sigma_0'(\xi)\frac{de_p}{d\xi} \tag{2.8}$$

It follows from the definition of R that

$$\frac{dR}{d\xi} = -J_0\frac{de_p}{d\xi} + q(\xi) \tag{2.9}$$

where $q(\xi)$ are terms which are independent of $de_p(\xi)$.

Using (2.8) and (2.9) and the definition of de_p, we have

$$\frac{de}{d\xi} = \frac{de_p}{d\xi} + J_0^{-1}\frac{d\sigma}{d\xi} = \frac{\sigma_0(\xi)}{J_0}\frac{d^2 e_p}{d\xi^2} + \sigma_0'(\xi)\frac{de_p}{d\xi} + \frac{q(\xi)}{J_0}$$

On scalar multiplying de by de_p, we get

$$d\xi = Q^{-1}\frac{(\sigma - R)\,de}{\sigma_0(\xi)}, \quad Q = \sigma_0'(\xi) + \frac{q(\xi)(\sigma - R)}{J_0\sigma_0(\xi)} \tag{2.10}$$

Since $d\xi \geqslant 0$, Eq.(2.10) is valid only if $n\,de \geqslant 0$, where $n = (\sigma - R)/\sigma_0(\xi)$.

The governing plastic deformation equation can now be written in the form

$$de = J_0^{-1}d\sigma + Q^{-1}n\,(n\,de)$$

where the last term is assumed to be different from zero only if $|n| = 1$ and $n\,de > 0$. When any of these conditions break down, plastic deformation ceases and an elastic process occurs.

Hence, the ETP relationships when $\chi = 1$ essentially reduce to the analogous relationships in flow theory in which the yield surface is described by expression (2.6), the displacement of the centre of the yield surface is determined by the vector R and the Odqvist parameter is used as the hardening parameter.

However, it is pertinent to note here that the theory of plasticity with $\chi = 1$ is close in its properties to the theory of plasticity proposed by Backhaus /13/

$$\sigma = a(\lambda)\frac{de_p}{d\lambda} + \int_0^\lambda G_1(\lambda - \lambda')\,G_2(\lambda')\frac{de_p}{d\lambda'}\,d\lambda'$$

which is not mentioned in /2-4/.

Such a development of a plasticity theory, which was initially considered as a theory without a yield surface, can hardly be considered as being coherent.

The condition $\chi = 1$ is excessively rigorous. It has been shown in /14/ that a well defined meaning can be ascribed to the parameter χ and, moreover, it may not only differ from unity but also vary during the course of the deformation process. Furthermore, it has been shown that the inequality $\chi < 1$ is essential for describing a number of effects which are observed during the complex deformation of metals /8, 9/.

3. So, a yield surface arises in ETP when $\chi = 1$. When $\chi < 1$, there is now, of course, no such surface but, for purposes of comparison with the classical theories of plasticity and, in particular, with flow theory or ETP when $\chi = 1$, the concept of a hypothetical flow surface can be introduced.

As is well-known, the yield surface is determined in experiments by making a certain assumption. Most frequently, the margin on the magnitude of the residual deformations is specified. The position, form and dimensions of the yield surface depend on the magnitude of the adopted margin. It is also possible to do the same in the case of ETP.

We shall say that a quasi-elastic deformation process from a certain state σ is realized, during loading, with an accuracy δ if

$$|de_p/ds| < \delta \tag{3.1}$$

We shall call the domain of the stress space Ω_δ at each point of which a quasi-elastic deformation process is realized under any load the domain of quasi-elasticity. We shall call the boundary of this domain $S_\delta = \partial\Omega_\delta$ the hypothetical yield surface. Its dimensions, form and position in the stress space therefore depend on the magnitude of the margin δ which is adopted.

Let us construct the hypothetical yield surface for a ETP defined by relationships (1.1), (1.3) with $\chi < 1$.

For the second term on the right-hand side of relationship (2.1), we introduce the notation $\mathbf{A}dz$. It is then possible to write that

$$\left|\frac{de_p}{ds}\right| = \frac{A_u}{J_0}\frac{dz}{ds}, \quad A_u = |\mathbf{A}|$$

On substituting (2.1) into the definition of dz (1.3), we get

$$f\frac{dz}{ds} = \left|(1-\chi)\frac{d\mathbf{e}}{ds} - \chi\frac{\mathbf{A}}{J_0}\frac{dz}{ds}\right|$$

On squaring this equation, we arrive at a quadratic equation for dz/ds, the root of which (the second root is constant and therefore discarded) can be written in the form

$$\frac{dz}{ds}(\gamma) = \frac{1-\chi}{f^2-\varphi^2}\left(-\varphi\cos\gamma + \sqrt{f^2-\varphi^2\sin^2\gamma}\right), \quad \varphi = \frac{\chi A_u}{J_0} \tag{3.2}$$

where γ is the angle between the vectors $d\mathbf{e}/ds$ and $\mathbf{A}$.

The quantity $\max_\gamma(dz/ds)$ is of interest since, if the inequality (3.1) is satisfied in the case of a $d\sigma$ which corresponds to the maximum dz/ds, then it will also hold for any other possible $d\sigma$.

It follows from (3.2) that dz/ds takes a minimum value when $\cos\gamma = -1$ and $\max_\gamma(dz/ds) = (1-\chi)/(f-\varphi)$. With such a value of dz/ds, the inequality (3.1) takes the form

$$A_n < (1+\chi\delta-\chi)^{-1}\,\delta J_0 f$$

It follows from the results of Sect.2 that

$$A_u = \left|\int_0^z J'(z-\eta)\,d\mathbf{e}(\eta)\right| = -\frac{J_0'}{J_0}|\sigma - \mathbf{R}|$$

and, hence, we finally obtain that the domain of quasi-elasticity is defined by the inequality

$$|\sigma - \mathbf{R}| < \sigma_\delta(\xi) = (1-\chi+\delta\chi)^{-1}\,\delta\sigma_0(\xi) \leqslant \sigma_0(\xi)$$

The hypothetical yield surface S_δ is correspondingly defined by the equation $|\sigma - \mathbf{R}| = \sigma_\delta(\xi)$. Hence, when $\chi < 1$ the hypothetical yield surface in ETP, like the previously introduced limiting surface, is defined by essentially the same laws. The properties of the ETP models for $\chi = 1$ and for $\chi < 1$ are quite different. When $\chi < 1$, the surface S_δ is a purely hypothetical concept and does not lead to a breakdown in the analytical nature of the theory. During deformation, the stress vector σ may pierce this surface, which is not accompanied by any singularities.

When $\chi < 1$, there is a plasticity layer in ETP in which the plastic deformations are substantial, that is, inequality (3.1) breaks down. This layer is defined by the inequalities

$$\sigma_\delta(\xi) < |\sigma - \mathbf{R}| < \sigma_0(\xi)$$

When $\chi \to 1$, the plasticity layer collapses into the yield surface. The structure and properties of the plasticity, layer been considered in greater detail in /15/.

In concluding, we would make one remark. In the arguments which have been presented above it was important that the kernel J was non-singular. If this is not the case then a yield surface also cannot exist.

In a number of papers on ETP the use of plasticity functionals in the form of (1.4) with a singular kernel J has been proposed and non-singular approximations of this kernel are used in actual calculations /17, 18/. It can be shown that, in this case, a "replacement" of the model actually takes place implicity and that, in fact, the version of ETP with $\chi < 1$ is being considered in the calculations.

In order to prove this, we rewrite the functional (1.4) in the form

$$d\sigma = J_0 d\mathbf{e}_p + \int_0^z J'(z-\eta)\,d\mathbf{e}_p(\eta)\,dz$$

By substituting the definition of $d\mathbf{e}_p$ from this, we obtain that the "present" modulus of elasticity is not E but $E_1 = \chi E$, where $\chi = J_0/(E+J_0) < 1$ and therefore

$$d\xi = |d\mathbf{e} - E^{-1}d\sigma| = |d\mathbf{e} - \chi E_1^{-1}d\sigma|$$

It should therefore be acknowledged that the proposal due to Valanis /2/: $dL = d\xi = |de - \chi E^{-1} d\sigma|$, $\chi < 1$ is the most successful of the simplest forms of writing the "new" internal time measure. Attempts to identify the parameter L with the Odqvist parameter reduce the possibilities of ETP. The case when $\chi < 1$, but $\chi \sim 1$ is of the greatest interest.

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